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PARAMETRIZATION OF DOMAINS IN $\widehat{\mathbf{C}}$: THE LOGARITHMIC DOMAINS

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ABSTRACT. We prove a generalization of Riemann's mapping theorem: Every n-fold connected domain in $\widehat{\mathbf{C}}$, whose boundary does not contain isolated points, is conformal equivalent to a logarithmic domain. The logarithmic domains are characterized by a Green's function consisting of a finite sum of logarithms.

1. INTRODUCTION

In this article we prove a generalization of Riemann's mapping theorem. We show that every n-fold connected domain in $\widehat{\mathbf{C}}$ $(n \geq 2)$ with the property that each boundary component consisting of more than one point is completely characterized up to conformity by its system of Green's parameters.

A system of Green's parameters of a domain D (Definition 3.2) is determined by $3n - 4$ real parameters. These parameters are given by the values of the Green's function of D and of a fixed branch of its conjugate harmonic function taken at their $n - 1$ critical points. This number of real parameters is the same as in the classical theory if we consider the free choice of the pole for the Green's function.

Using this parametrization for planar domains we can prove the main result of this article, Theorem 5.1: The logarithmic domains form a "canonical family" of domains; by this we mean that they form a complete set of representatives of the equivalence classes with respect to conformity of n-fold connected domains in $\widehat{\mathbf{C}}$.

The logarithmic domains are those domains in $\mathbf{C}$, whose Green's function consists of a finite sum of logarithms: $n + 1$ sum terms for n-fold connected domains (Definition 4.1).

The known results in this direction are summarized in a theorem given in Walsh [1]:

Theorem. *Let R be an infinite region with finite boundary B possessing a Green's function $G(x, y)$ with pole at infinity. An arbitrary level locus B_μ can be approximated by a lemniscate, in the sense that if δ is arbitrary, $0 < \delta < \mu$,*

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there exists a lemniscate λ whose poles are exterior to $R_{\mu-\delta}$, such that the infinite region bounded by λ contains $R_{\mu+\lambda}$ and is contained in $R_{\lambda-\mu}$. If B_μ consists of m disjoint Jordan curves, so also does λ.

Note that in this theorem the number of singularities of the lemniscate λ is not bounded.

We conclude by giving some applications to the main result.

2. Basic definitions

Let $\mathscr{D}$ be the set of all n-fold connected domains in $\widehat{\mathbf{C}}$ $(n \geq 2)$ with the property that each boundary component consists of more than one point. This set can be divided into equivalence classes regarding conformity. We denote by D_n a set of representatives of these equivalence classes such that the boundary of each $D \in D_n$ consists of the union of n analytic curves. The existence of such a system is well known.

We denote by $G(\cdot, z_0)$ and $H(\cdot, z_0)$, $z_0 \in D$, $z_0 \neq \infty$ the Green's function and its conjugate harmonic function. $H(\cdot, z_0)$ is multiple valued and its definition is not unique.

As in Walsh [2] we define for any fixed $H(\cdot, z_0)$ the trajectories

$$t_a := \{z \in D \mid e^{-iH(z, z_0)} = e^{-ia}, \ a \in \mathbf{R}\}.$$

the level lines

$$h_s := \{z \in D \mid e^{-G(z, z_0)} = e^{-s}, \ s \in \mathbf{R}\},$$

and the set of critical points of D

$$C(D) := \{w \in D \mid \operatorname{grad} G(z, z_0)|_w = 0\}.$$

A critical point w is of order k_w if

$$\left.\frac{\partial^l G(z, z_0)}{\partial z^l}\right|_w = 0,$$

for $l = 1, \dots, k_w$ and

$$\left.\frac{\partial^{k_w+1} G(z, z_0)}{\partial z^{k_w+1}}\right|_w \neq 0.$$

Here $\partial G(z, z_0)/\partial z$ means the Wirtinger derivative of $G(\cdot, z_0)$. One easily checks that $\sum_{w \in C(D)} k_w = n - 1$.

We say that a trajectory t_a leads from z_0 to $\hat{z}$ if the set

$$(\{z \in D \mid e^{-iH(z, z_0)} = e^{-ia}\} \cap \{z \in D \mid G(z, z_0) \geq G(\hat{z}, z_0)\}) \cup \{\hat{z}, z_0\}$$

is connected. There are at least $k_w + 1$ trajectories from z_0 to a critical point w of order k_w. It may happen that one trajectory leads to different critical points, say $w_1, \dots, w_l$, $l \geq 2$. Choose w_1 in such a way that $G(w_1, z_0) \geq G(w_j, z_0)$, $j = 2, \dots, l$. In this case at least 2 trajectories from z_0 to the critical points w_j, $j > 1$, are equal in a suitable neighbourhood of w_j. For the following we will not distinguish between these trajectories. With respect to this identification there are exactly $k_w + 1$ trajectories from z_0 to w for all $w \in C(D)$. In Walsh one can find a detailed description of the trajectories and critical points.

For a domain $D \in D_n$ we denote by $\Upsilon(D)$ the following set:

$$\Upsilon(D) := \left\{ c_k \in (0, 1) \mid c_k = \frac{1}{2\pi} \int_{\gamma_k} \frac{\partial G(z, z_0)}{\partial n_z} d\sigma_z \right\}_{k=1, \dots, n} .$$

The quantity c_k is the harmonic measure of the boundary component γ_k.

A point $\rho \in D \backslash C(D)$ is a regular point, if the set

$$T := \{z \in D\backslash C(D) \mid H(z, z_0) = H(\rho, z_0)\} \cap \{z \in D\backslash C(D) \mid G(z, z_0) \geq G(\rho, z_0)\}$$

is connected for a suitable branch of H. We denote by $R(D)$ the set of regular points of D. $R(D)$ is a dense open set in D and $R(D) \cup \{z_0\}$ is simply connected.

For every $D \in D_n$ we define the holomorphic function φ on a suitable neighbourhood U of z_0 by

$$\varphi : U \to \mathbf{K} := \{z \in \mathbf{C} \mid |z| \leq 1\}, \qquad z \mapsto e^{-G(z, z_0) - iH(z, z_0)}.$$

For the sake of definiteness we make the definition of H and hence of φ unique, requiring for the pole of φ^{-1} at 0 to be simple with positive residue.

The function φ defined on a neighbourhood U of z_0 can be uniquely extended to $R(D)$ by its analytic continuation along the trajectories defined on U and joining points in $R(D)$. To define φ for all points of D we proceed as follows: For $z \in D \backslash R(D)$ we define

$$\varphi(z) := \{\varphi_j(z)\}_j := \left\{ \lim_{k\to\infty} \varphi(z_k) \mid z_k \in R(D), \lim_{k\to\infty} z_k = z \right\},$$

where, when necessary we use appropriated subsequences of $(z_k)_{k\in\mathbf{N}}$ to assure the convergence of $(\varphi(z_k))_{k\in\mathbf{N}}$.

For $z \in D\backslash(R(D) \cup C(D))$ the set $\{\varphi_j(z)\}_j$ consists of two points. For $w \in C(D)$ it consists of at least two points and is finite. Further for $z \in D\backslash R(D)$ we have for the values of $\varphi(z)$

$$\arg \varphi_j(z) - \arg \varphi_i(z) = 2\pi \sum_{k\in S} c_k ,$$

where S is some subset of $\{1, \dots, n\}$ and $c_k \in \Upsilon(D)$.

3. The Green's parameters of a domain $D \in D_n$

Definition 3.1. Let $w \in C(D)$ be of order k_w. A $(k_w + 1)$-tuple $\varphi_{[w]} := (\varphi_1(w), \dots, \varphi_{k_w+1}(w))$ of values of φ in w is called a tuple of Green's parameters for w, if the sets

$$\{z \in D \mid \arg \varphi(z) = \arg \varphi_l(w), G(z, z_0) \geq G(w, z_0)\}_{l=1, \dots, k_w+1}$$

are different trajectories in every neighbourhood of w.

Remark. Due to the ambiguity concerning the trajectories as discussed in §2, there could exist more than one tuple of Green's parameters for a critical point. However, this ambiguity has no importance in our work.

Definition 3.2. The set $\varphi_{[D]} := \{\varphi_{[w]}\}_{w\in C(D)}$ is called a system of Green's parameters for (D, z_0), if every tuple $\varphi_{[w]}$ is a tuple of Green's parameters for w.

We now prove that the systems of Green's parameters completely characterize the elements of D_n. To this end we need some preliminary work.

Definition 3.3. Two points $x, y \in \mathbf{K} := \{z \in \mathbf{C} \,|\, |z| \le 1\}$ with $0 \le \arg x \le \arg y < 2\pi$ are adjoining if for each $w \in C(D)$ with $|\varphi(w)| < \min\{|x|, |y|\}$ the interval $[\arg x, \arg y]$ either contains all or none of the arguments of the components of $\varphi_{[w]}(\arg\colon \mathbf{K} \to [0, 2\pi))$. We call a sequence of points in $\mathbf{K}$, $(x_j)_{j\in\mathbf{N}}$ adjoining, if there is an $N > 0$ such that the elements x_j, $j > N$, are pairwise adjoining.

Remark. We denote the interval between the arguments of the components $\varphi_l(u)$ and $\varphi_m(u)$ of $\varphi_{[u]}$, $u \in C(D)$ by

$$I_{lm}(u) := \{\alpha \in [0, 2\pi) \mid \arg\varphi_l(u) < \alpha < \arg\varphi_m(u)\}.$$

For $u, v \in C(D)$ and $1 \le l, m \le (k_u + 1)$, $1 \le i, j \ge (k_v + 1)$ we have one of the three possibilities: $I_{lm}(u) \cap I_{ij}(v) = \emptyset$ or $I_{lm}(u) \cap I_{ij}(v) = I_{lm}(u)$ or $I_{lm}(u) \cap I_{ij}(v) = I_{ij}(v)$.

Lemma 3.1. *A sequence $(\zeta_j)_{j\in\mathbf{N}}$, $\zeta_j \in R(D)$ converges to a point in $D\backslash(R(D)\cup C(D))$ if and only if the sequence $(\varphi(\zeta_j))_{j\in\mathbf{N}}$ is adjoining and its accumulation points are in the set $\{\varphi_1(\zeta), \varphi_2(\zeta)\}$ for some $\zeta \in D\backslash(R(D)\cup C(D))$.*

Proof. (A) Let $(\zeta_j)_{j\in\mathbf{N}}$ converge to ζ with $\varphi(\zeta) = \{\varphi_1(\zeta), \varphi_2(\zeta)\}$. Then the sequence $(\varphi(\zeta_j))_{j\in\mathbf{N}}$ has its accumulation points in the set $\{\varphi_1(\zeta), \varphi_2(\zeta)\}$. Since $(\zeta_j)_{j\in\mathbf{N}}$ converges and $|\varphi|$ is continuous we have

$$\lim_{j\to\infty}(|\varphi(\zeta_j)|) = |\varphi_1(\zeta)| = |\varphi_2(\zeta)|.$$

Hence in order to prove the adjointness of $(\varphi(\zeta_j))_{j\in\mathbf{N}}$ we only need to consider the critical points in

$$C_\zeta(D) := \{w \in C(D) \mid |\varphi(w)| < |\varphi_1(\zeta)|\}.$$

We define the following sets:

$$U_1 := \{z \in R(D) \mid \arg\varphi_1(\zeta) < \arg\varphi(z) < \arg\varphi_2(\zeta)\},$$
$$U_2 := \{z \in R(D) \mid \arg\varphi_2(\zeta) < \arg\varphi(z) < \arg\varphi_1(\zeta) + 2\pi\}.$$

The convergence of $(\zeta_j)_{j\in\mathbf{N}}$ implies the existence of an $N_1 > 0$ such that either $\zeta_j \in U_1$ or $\zeta_j \in U_2$ for $j > N_1$. By the remark above there is an $N_2 > N_1$ such that the property "adjoining" with respect to the critical points w in $C_\zeta(D) \cap t_{\arg\varphi_1(\zeta)} \cap t_{\arg\varphi_2(\zeta)}$ is fulfilled. The other critical points of $C_\zeta(D)$

lie either in $\overline{U}_1$ or $\overline{U}_2$. Again due to the remark there exists an $N > N_2$ such that the property "adjoining" with respect to these critical points is also fulfilled.

(B) Let $(\varphi(\zeta_j))_{j\in\mathbf{N}}$ be an adjoining sequence with accumulation points lying in the set $\{\varphi_1(\zeta), \varphi_2(\zeta)\}$, with $\zeta \in D\backslash(R(D)\cup C(D))$. the property "adjoining" assures the existence of $N > 0$ such that for $j \geq N$ all ζ_j lie in a compact set $\overline{U}$, where $\overline{U}$ contains exactly one point η with $\lim_{j\to\infty}\varphi(\zeta_j) = \varphi(\eta)$. The compactness of $\overline{U}$ and the continuity of φ then imply $\lim_{j\to\infty}\zeta_j = \eta$.

Proposition 3.1. *Let M and N be two domains in $\mathscr{D}$. If M and N have the same system of Green's parameters, then they are conformal equivalent.*

Proof. The map

$$\psi: R(M) \to R(N), \qquad z \mapsto \varphi_N^{-1} \circ \varphi_M(z)$$

is conformal on $R(M)$. By the preceding lemma ψ can be continuously extended to a map from $R(M)\backslash C(M)$ to $R(N)\backslash C(N)$. By the reflection theorem this extension is again conformal. Further, since the functions φ_M and φ_M^{-1} are bounded ψ can be extended to a conformal map from M to N.

Definition 3.4. Two domains D and $E \in D_n$ are c-equivalent, if there is a continuous bijective map $f: \mathbf{K} \to \mathbf{K}$, $f|_{\partial\mathbf{K}}$ homotopic to the identity such that $\varphi_D[R(D)] = f \circ \varphi_E[R(E)]$.

Definition 3.5. A c-equivalence class is called open, if it only contains domains whose systems of Green's parameters are such that the arguments of its components are pairwise different. We call a surjective continuous function $f: \mathbf{K} \to \mathbf{K}$ applicable if it is a uniform limit of bijective continuous maps, which are homotopic to the identity, when restricted to $\partial\mathbf{K}$.

Definition 3.6. A domain D is in the closure of the open c-equivalence class $\mathfrak{F}$ if there is an applicable function f and a domain $\widehat{D} \in \mathfrak{F}$ such that $\varphi_D[R(D)] = f \circ \varphi_{\widehat{D}}[R(\widehat{D})]$. Note that every domain $D \in D_n$ lies in the closure of an open c-equivalence class.

Definition 3.7. A tuple $\phi := ((\phi_1^j, \phi_2^j))_{j=1,\dots,n-1} \in (\mathbf{C}^2)^{n-1}$ is of parameteric type if

(1) $|\phi_l^j| < 1$ for all $l = 1, 2$ and $j = 1, \dots, n-1$.

(2) $\arg\phi_l^i \neq \arg\phi_k^j$ for all $l, k = 1, 2$ and $j, i = 1, \dots, n-1$.

(3) For the intervals $I_j := (\arg\phi_1^j, \arg\phi_2^j)$ the following holds: $I_j \cap I_i = \varnothing$, or $I_j \cap I_i = I_j$, or $I_j \cap I_i = I_i$, where $j, i = 1, \dots, n-1$.

Remark. If $D \in D_n$ lies in an open c-equivalence class, then its system of Green's parameters is a parametric type tuple.

The set $\{1, \dots, n-1\}$ can be partially ordered by the inclusion property of the intervals I_i's: $(i < j$ if $I_i \subset I_j$, $I_i \neq I_j)$. We call the numbers $c_j := \lambda(I_j) - \lambda(\bigcup_{i<j} I_i)$ where λ is the usual measure in $\partial\mathbf{K}$ the residues of ϕ. For the system of Green's parameters of a domain in an open c-equivalence

class the c_j 's correspond to the harmonic measures of the boundary components of the domain. Denote by $R(\phi)$ the complement in $\mathbf{K}$ of the set

$$\{z \in \mathbf{K} \mid z = a\phi_l^j \,,\, a \in [1\,,\infty)\,,\, l = 1\,,2 \text{ and } j = 1\,,\ldots\,,n-1\}.$$

We extend the c-equivalence to the parametric type tuples.

Definition 3.8. Let ϕ and η be two parametric type tupels. We call ϕ and η c-equivalent if there is a continuous bijective function $f:\mathbf{K} \to \mathbf{K}$ which is homotopic to the identity on $\partial\mathbf{K}$ such that $R(\phi) = f[R(\eta)]$.

4. The logarithmic domains

We denote by Δ_n, $n \geq 2$, the set of all functions l_n,

$$l_n:\mathbf{C} \to \mathbf{C}, \qquad z \mapsto -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n} c_j \log|z-\xi_j|,$$

with $0 < c_j < 1$ for $j = 1\,,\ldots\,,n$ and $\sum_{j=1}^{n} c_j = 1$, as well as $\xi_j \in \mathbf{C}$ with $\xi_j \neq \xi_l$ for $j \neq l$ and $\xi_j \neq 0\,,1$ for $j = 2\,,\ldots\,,n$.

For each $l_n \in \Delta_n$ there exists a constant $a \in \mathbf{R}$ such that the set

$$D_a := \{z \in \mathbf{C} \mid l_n(z) > a\}$$

is an n-fold connected domain, where its function φ as defined in §2 is given by

$$\varphi(z) = \alpha z(z-1)^{-c_1} \prod_{j=2}^{n} (z-\xi_j)^{-c_j} e^a, \qquad |\alpha| = 1.$$

Note that α is defined such that the pole in 0 of φ^{-1} is simple with positive residue.

Definition 4.1. An n-fold connected domain $L \subset \mathbf{C}$ is called logarithmic if $L = \{z \in \mathbf{C} \mid l_n(z) > 0\}$ for some $l_n \in \Delta_n$.

Lemma 4.1. *Every n-fold connected domain $D_a = \{z \in \mathbf{C} \mid l_n(z) > a\}$ is conformal equivalent to a logarithmic domain.*

Proof. Define a conformal map $f:\widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ by $f(0) = 0$, $f(1) = 1$, and $f(u) = \infty$, for some u with $l_n(u) = a$. Then the Green's function of the domain $f[D_a]$ is

$$\widehat{l}_n(z) = -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n} c_j \log|z - f(\xi_j)|.$$

The following definition of the system of Green's parameters for the elements of Δ_n is by the preceding lemma compatible with Definition 3.2.

Definition 4.2. Let $\varphi_{[D_a]} = \{\varphi_{[w]}\}_{w\in C(D_a)}$ be a system of Green's parameters of D_a. The set

$$\varphi_{[l_n]} := \{\varphi_{[w]}e^{-a}\}_{w\in C(D_a)}$$

is called a system of Green's parameters of l_n.

A function $l_n \in \Delta_n$ belongs to a c-equivalence class of parametric type tuples $\mathfrak{R}$ if $\varphi_{[l_n]}$ belongs to $\mathfrak{R}$. Two functions, l_n and $\widehat{l}_n$ belong to the same c-equivalence class, if $\varphi_{[l_n]}$ and $\varphi_{[\widehat{l}_n]}$ are c-equivalent.

Let $\mathfrak{R}$ be a fixed open c-equivalent class of functions $l_n \in \Delta_n$. Consider a sequence $(l_n^k)_{k\in\mathbf{N}}$ of function in $\mathfrak{R}$ with

$$\lim_{k\to\infty} \xi_j^k = \eta_j \in \widehat{\mathbf{C}}, \qquad j = 2, \dots, n.$$

Lemma 4.2. (1) *If one* $\eta_j = 0$, *then for every* $k \in \mathbf{N}$ *there is at least one critical point* w^k *of* l_n^k *such that the sequence* $l_n^k(w^k)$ *tends to infinity.*

(2) *If for one* $j > 2$, $\eta_j = 1$ *or* $\eta_j = \eta_i$ *for* $j \neq i$ *and* $\eta_j \neq \infty\ \forall j$, *then for every* $k \in \mathbf{N}$ *there is at least one critical point* w^k *of* l_n^k *such that the sequence* $|l_n^k(w^k)|$ *tends to infinity.*

(3) *If for one* $j > 2$, $\eta_j = \infty$ *and* $\eta_j \neq 0\ \forall j$, *then for every* $k \in \mathbf{N}$ *there is at least one critical point* w^k *of* l_n^k *such that the sequence* $|l_n^k(w^k)|$ *tends to infinity.*

(4) *Let* $K > 0 \in \mathbf{R}$, $\eta_i \neq 0, 1, \infty$ *for all* $j > 2$ *with* $\eta_j \neq \eta_i$ *for* $j \neq i$. *If* $\sum_{j=2}^n c_j\eta_j + c_1 = 0$, *and if for all* $k > K$, $\sum_{j=2}^n c_j\xi_j^k + c_1 \neq 0$, *then there exists a sequence* $(w^k)_{k\in\mathbf{N}}$ *of critical points of* l_n^k *such that* $\lim_{k\to\infty} w^k = \infty$, *hence* $\lim_{k\to\infty} l_n^k(w^k) = 0$.

Proof. (ad 1) Renumber the ξ_j^k in such a way that $\lim_{k\to\infty} \xi_j^k = 0$ for $j \le m_1$, $\lim_{k\to\infty} \xi_j^k \in \mathbf{C}\backslash\{0\}$ for $m_1 < j \le m_2$ and $\lim_{k\to\infty} \xi_j^k = \infty$ for $j > m_2$. The sequence of functions

$$F_{m_2}^k(z) := -\log|z| + c_1\log|z-1| + \sum_{2\le j\le m_2} c_j\log|z-\xi_j^k|$$

and the sequence of its derivatives converge uniformly in every compact set not containing the points $\{0, 1, \eta_j, \ j = 2, \dots, m_2\}$ to the function

$$F_{m_2}(z) := -\log|z|\left(1 - \sum_{2\le j\le m_1} c_j\right) + c_1\log|z-1| + \sum_{m_1<j\le m_2} c_j\log|z-\eta_j|$$

or to its derivative respectively. This means that for a sufficiently small $\varepsilon > 0$ there is a $K > 0 \in \mathbf{R}$ such that for all $k > K$

(a) $\xi_j^k \in U_\varepsilon(0)$ for $2 \le j \le m_1$ and $\xi_j^k \notin U_\varepsilon(0)$ for $j > m_1$.

(b)

$$\left|\left(\frac{\partial F^k_{m_2}(re^{i\alpha})}{\partial r} - \frac{\partial F_{m_2}(re^{i\alpha})}{\partial r}\right)\bigg|_\varepsilon\right| < \varepsilon \quad \text{and} \quad \frac{\partial F_{m_2}(re^{i\alpha})}{\partial r}\bigg|_\varepsilon < 0$$

for all $\alpha \in [0, 2\pi)$.

(c)

$$\left|\sum_{j=m_2+1}^{n} \frac{c_j}{z - \xi^k_j}\right| < \varepsilon$$

for all $z \in U_\varepsilon(0)$.

(d) $|F^k_{m_2}(z) - F_{m_2}(z)| < \varepsilon$ for all $z \in \partial U_\varepsilon(0)$.

Since the function $\partial F_{m_2}(z)/\partial z$ is holomorphic, by (a), (b), (c), and the theorem of Rouché there is at least one critical point w^k in $U_\varepsilon(0)$ of the function l^k_n with

$$l^k_n(w^k) > \min_{z \in \partial U_\varepsilon(0)} \left(-\log|z| + c_1 \log|z-1| + \sum_{j=2}^{m_2} c_j \log|z - \xi^k_j|\right).$$

As $\varepsilon \to \infty$, $l^k_n(w^k)$ tends to infinity according to (d).

(ad 2) Without loss of generality we just prove this lemma for $\eta_n = \eta_{n-1}$ and $\eta_j \neq 0\ \forall j$. By assumption $\eta_j \neq \infty\ \forall j$, hence the sequence of functions $l^k_n(z)$ and the sequence of its derivatives converge uniformly in every compact set not containing the points $\{0, 1, \eta_j,\ j = 2, \dots, n-1\}$ to the function

$$l_n(z) = -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n-2} c_j \log|z - \eta_j| + (c_{n-1} + c_n)\log|z - \eta_n|$$

or to its derivative respectively. This means that for a sufficiently small $\varepsilon > 0$ there is a $K > 0 \in \mathbf{R}$ such that for all $k > K$

(a) $\xi^k_j \in U_\varepsilon(\eta_n)$ for at least two j's.

(b)

$$\left|\left(\frac{\partial l^k_n(re^{i\alpha} + \eta_n)}{\partial r} - \frac{\partial l_n(re^{i\alpha} + \eta_n)}{\partial r}\right)\bigg|_\varepsilon\right| < \varepsilon$$

for all $\alpha \in [0, 2\pi)$.

(c) $|l_n(z) - l^k_n(z)| < \varepsilon$ for all $z \in \partial U_\varepsilon(\eta)$.

Since the function $\partial l_n(z)/\partial z$ is holomorphic, by (a), (b), and the theorem of Rouché there is a critical point w^k in $U_\varepsilon(\eta_n)$ of the function l^k_n. For sufficiently small $\varepsilon > 0$ we have

$$l_n(w^k) \leq \max_{z \in \partial U_\varepsilon(\eta)} l_n(z).$$

Hence by (c) we have that $|l^k_n(w^k)|$ tends to infinity as $\varepsilon \to 0$.

(ad 3) We renumber η_j such that $\eta_j \in \mathbf{C}$ for $2 \le j \le m$ and $\eta_j = \infty$ for $j > m$. The sequence of functions

$$F_m^k(z) := -\log|z| + c_1 \log|z-1| + \sum_{j=1}^{m} c_j \log|z - \xi_j^k|$$

and the sequence of its derivatives converge uniformly in every compact set not containing the points $\{0, 1, \eta_j, \ j = 2, \dots, m\}$ to the function

$$F_m(z) := -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{m} c_j \log|z - \eta_j|$$

or to its derivative respectively. Hence there exists a sequence of critical points $\widehat{w}^k$ of the functions F_m^k which converges to $w \in \mathbf{C}$, $w \neq 0$. Since the Wirtinger derivative of l_n^k converges in the above-mentioned compact sets to the holomorphic function $\partial F_m/\partial z$, by the theorem of Rouché there exists a sequence of critical points w^k of l_n^k which converges to w too. $\lim_{k\to\infty} |l_n^k| = \infty$ yields $\lim_{k\to\infty} |l_n^k(w^k)| = \infty$.

(ad 4) Since the w^k's are the solutions of

$$\frac{\partial l_n^k(z)}{\partial z} = \frac{-1}{z} + \frac{c_1}{z-1} + \sum_{j=2}^{n} \frac{c_j}{z - \xi_j^k} = 0$$

there are exactly $n-1$ critical points of l_n^k with order taken into account. By the assumption

$$\lim_{k\to\infty} l_n^k = l_n = -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n} c_j \log|z - \eta_j|$$

exists. The critical points of l_n are the solutions of $\partial l_n(z)/\partial z = 0$. Since $\sum_{j=2}^{n} c_j \eta_j + c_1 = 0$, we have at most $n-2$ critical points with order taken into account of l_n. This yields, by the theorem of Rouché, the existence of a sequence of critical points w^k of l_n^k with $\lim_{k\to\infty} w^k = \infty$.

Lemma 4.3. *Let $\mathfrak{R}$ be a c-equivalence class of parametric type tuples and $\phi \in \mathfrak{R}$. Then $\mathfrak{R}$ contains a function l_n with*

$$l_n(z) = -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n} c_j \log|z - \xi_j|$$

such that the c_j's correspond to the residues of ϕ.

Proof. We prove the lemma by induction. For $n = 2$ the function

$$l_2 = -\log|z| + c_1 \log|z-1| + c_2 \log|z-2|$$

has the desired property.

Let $n > 2$. By renumbering the tuple ϕ we can obtain $\arg\phi_l^j \notin (\arg\phi_1^{n-1}, \arg\phi_2^{n-1})$ for all $l = 1, 2$ and $j = 1, \dots, n-2$. Denote by $\mathfrak{R}_{n-1}$ the c-equivalence class of the tuple

$$((\arg\phi_1^1, \arg\phi_2^1), \dots, (\arg\phi_1^{n-2}, \arg\phi_2^{n-2})).$$

By induction $\mathfrak{R}_{n-1}$ contains a function l_{n-1},

$$l_{n-1}(z) = -\log|z| + c_1' \log|z-1| + \sum_{j=2}^{n-1} c_j' \log|z - \xi_j|.$$

According to the construction of $\mathfrak{R}_{n-1}$ we can assume without loss of generality that $c_j' = c_j$ for $j = 1, \dots, n-2$ and $c_{n-1}' = c_n + c_{n-1}$. Denote by w_j, $j = 1, \dots, n-2$, the critical points of l_{n-1} and by $(\varphi_{[w_1]}, \dots, \varphi_{[w_{n-2}]})$ the system of its Green's parameters. Consider for $r > 0 \in \mathbf{R}$ and $\alpha \in [0, 2\pi)$ the functions

$$\begin{aligned} l_n^\alpha(z) = & -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n-1} c_j \log|z - \xi_j| \\ & + c_n \log|z - \xi_{n-1} + re^{i\alpha}|, \end{aligned}$$

and denote by w_j^α its critical points. For every $\varepsilon > 0$ there is an $r > 0$ such that for all $\alpha \in [0, 2\pi)$

(a) $|w_j - w_j^\alpha| < \varepsilon$.

(b) $|\varphi_l(w_j) - \varphi_l^\alpha(w_j^\alpha)| < \varepsilon$, for an adequate numbering of the critical points w_j^α, $j = 1, \dots, n-2$, $l = 1, 2$..

Hence for sufficiently small $\varepsilon > 0$ the tuple $(\varphi^\alpha_{[w_1^\alpha]}, \dots, \varphi_{[w_{n-2}^\alpha]})$ belongs for every $\alpha \in [0, 2\pi)$ to $\mathfrak{R}_{n-1}$. Choose now α_0 such that $\arg\phi_1^{n-1} = \arg\varphi_1^{\alpha_0}(w_{n-1}^{\alpha_0})$. For sufficiently small $\varepsilon > 0$ then $l_n^{\alpha_0}$ lies in $\mathfrak{R}$ and has the desired c_j's.

Denote by O the following set:

$$O := \Big\{ \xi = (\xi_2, \dots, \xi_n) \in \mathbf{C}^{n-1} | \xi_j \neq 0, 1\ \forall j, \\ \xi_j \neq \xi_i\ \forall j \neq i, \text{ and } \sum_{j=2}^{n} c_j \xi_j + c_1 \neq 0 \Big\}.$$

Lemma 4.4. *Let w_l, $l = 1, \dots, n-1$, be the critical points of*

$$l_n(z) = -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n} c_j \log|z - \xi_j|.$$

Then the following map is continuous:

$$W: O \to \mathbf{C}^{n-1}, \qquad \xi \mapsto (\sigma_1(w_1, \dots, w_{n-1}), \dots, \sigma_{n-1}(w_1, \dots, w_{n-1})),$$

where σ_k denotes the symmetrical polynomial of order k.

Proof. Since the w_k 's are the solutions of

$$\frac{-1}{z} + \frac{c_1}{z-1} + \sum_{j=2}^{n} \frac{c_j}{z-\xi_j} = 0$$

we get

$$\sigma_{n-1}(w_1, \dots, w_{n-1}) = A[\sigma_n(\xi_1, \dots, \xi_n)],$$

with $A \in \mathbf{C}$ and $\xi_1 = 1$. Further, for $k \le n-2$ we get

$$\sigma_k(w_1, \dots, w_{n-1}) = A\left[\sigma_{k+1}(\xi_1, \dots, \xi_n) - \sum_{i=1}^{n} c_i \sigma_{k+1}(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_n)\right],$$

where

$$\sigma_k(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_n) = \sigma_k(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n).$$

This shows the continuity of W.

5. Characterization of n-fold connected domains in $\widehat{\mathbf{C}}$

Let $\mathfrak{R}$ be a c-equivalence class of parametric type tuples. Denote by $\mathfrak{R}(c_1, \dots, c_n)$ the subset of $\mathfrak{R}$ whose elements have residues equal to the c_j 's. Note that $\mathfrak{R}(c_1, \dots, c_n)$ is connected.

By Lemma 4.3 there exists at least one function l_n^0,

$$l_n^0(z) = -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n} c_j \log|z - \xi_j^0|$$

which belongs to $\mathfrak{R}(c_1, \dots, c_n)$. Hence the set

$$M := \left\{(\xi_2, \dots, \xi_n) \in O \mid l_n(z) = -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n} c_j \log|z-\xi_j| \in \mathfrak{R}(c_1, \dots, c_n)\right\},$$

where O, defined at the end of §4, is not empty.

For every $l_n \in \mathfrak{R}(c_1, \dots, c_n)$ there exists by definition of the c-equivalence, a bijective continuous function $f: \mathbf{K} \to \mathbf{K}$, $f|_{\partial\mathbf{K}}$ homotopic to the identity such that $\varphi[R(L)] = f \circ \varphi^0[R(L^0)]$, where L, L^0 are the n-fold connected domains determined by l_n and l_n^0 respectively, and where φ^0 denotes the function φ corresponding to L^0. Let $w_1, \dots, w_{n-1}$ be the critical points of the function l_n,

$$l_n(z) = -\log|z| + \sum_{j=1}^{n} c_j \log|z - \xi_j|,$$

with $\xi_1 = 1$ and $(\xi_2, \dots, \xi_n) \in M$.

Set

$$\widehat{\varphi}(w_i) := f(\varphi_1^0(w_i^0)) \in \{\varphi_1(w_i), \varphi_2(w_i)\}.$$

Lemma 5.1. *The function*

$$\mathbf{V}: M \to \mathbf{C}^{n-1}, \qquad (\xi_2, \ldots, \xi_n) \mapsto (\widehat{\varphi}(w_1), \ldots, \widehat{\varphi}(w_{n-1}))$$

is open.

Proof. By explicit calculation we obtain $|\det(\partial \mathbf{V}_k / \partial \xi_j)| \neq 0$. This yields the openness of $\mathbf{V}$.

Lemma 5.2. *If the sequence of systems of Green's parameters* $(\varphi_{[l_n^k]})_{k \in \mathbf{N}}$ *of the functions* l_n^k,

$$l_n^k(z) = -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n} c_j \log|z - \xi_j^k|,$$

all belonging to the c-equivalence class $\mathfrak{R}$, *converges to a parametric type tuple* ϕ, *then there exists a function* l_n *with* $\varphi_{[l_n]} = \phi$.

Proof. Without loss of generality we can assume that ξ_j^k converges to ξ_j for all j. From Lemma 4.2 and Lemma 4.4 it follows the existence of $\lim_{k\to\infty} l_n^k = l_n$,

$$l_n(z) = -\log|z| + c_1 \log|z-1| + \sum_{j=2}^{n} c_j \log|z - \xi_j|,$$

with $\xi_j \in \mathbf{C}$, $\xi_j \neq \xi_l$ for $j \neq l$ and $\xi_j \neq 0, 1$ for $j = 2, \ldots, n$ such that the system of Green's parameters $(\varphi_{[l_n^k]})_{k\in\mathbf{N}}$ converges to $\varphi_{[l_n]}$.

Proposition 5.1. *The following map is surjective*:

$$\mathbf{F}: M \to \mathfrak{R}(c_1, \ldots, c_n), \qquad (\xi_2, \ldots, \xi_n) \mapsto \varphi_{[l_n]},$$

where l_n *is determined by the* c_j*'s and the* ξ_j*'s.*

Proof. The values $\widehat{\varphi}(w_i)$, $i = 1, \ldots, n-1$, determine $\varphi_{[l_n]}$. Since M is not empty, Lemma 5.1 and Lemma 5.2 together with the connectivity of $\mathfrak{R}(c_1, \ldots, c_n)$ yield the surjectivity of $\mathbf{F}$.

Proposition 5.2. *Let* $D \in D_n$ *lie in the closure of an open c-equivalence class* $\mathfrak{T}$. *Then there exists a function* l_n *with the same system of Green's parameters,* $\varphi_{[l_n]} = \varphi_{[D]}$.

Proof. Since $\mathfrak{T}$ is open, the system of Green's parameters of the elements of $\mathfrak{T}$ lie in a c-equivalence class of parametric type tuples $\mathfrak{R}$. By the definition of the closure of an open c-equivalence class (Definition 3.6) there exists a domain $\widehat{D}$ with $\varphi_{[\widehat{D}]} \in \mathfrak{R}$ and a sequence of bijective continuous functions $f^k: \mathbf{K} \to \mathbf{K}$, $f^k|_{\partial\mathbf{K}}$ homotopic to the identity with uniform limit equal to the surjective function $f: \mathbf{K} \to \mathbf{K}$ such that

$$\varphi_D[R(D)] = \lim_{k\to\infty} f^k(\varphi_{[\widehat{D}]}[R(\widehat{D})]) = f \circ \varphi_{\widehat{D}}[R(\widehat{D})].$$

By Proposition 5.1 there exists for every k a function $l_n^k \in \mathfrak{R}$ with

$$\varphi_{[l_n^k]} = (\{f^k(\varphi_{\widehat{D}1}(\widehat{w}_1)), f^k(\varphi_{\widehat{D}2}(\widehat{w}_1))\}, \dots, \{f^k(\varphi_{\widehat{D}1}(\widehat{w}_{n-1})), f^k(\varphi_{\widehat{D}2}(\widehat{w}_{n-1}))\}).$$

According to Lemma 4.2 there exists for every sequence $(l_n^k)_{k\in\mathbf{N}}$ a convergent subsequence $(l_n^{k_\nu})_{k_\nu\in\mathbf{N}}$ with $\lim_{k_\nu\to\infty} l_n^{k_\nu} = l_n$, and $\varphi_{[l_n]} = \lim_{k\to\infty} \varphi_{[l_n^k]}$. But by construction we have $\lim_{k\to\infty} \varphi_{[l_n^k]} = \varphi_{[D]}$. This completes the proof.

Theorem 5.1. *Every n-fold connected domain $D \in \mathscr{D}$ is conformal equivalent to a logarithmic domain.*

Proof. Without loss of generality we can assume that the boundary consists of analytic curves. According to Proposition 3.1 D is completely characterized up to conformity by $\varphi_{[D]}$. By Proposition 5.2 there exists a function l_n whose system of Green's parameters is equal to $\varphi_{[D]}$. Hence by Lemma 4.1 there exists a logarithmic domain which is conformal equivalent to D.

6. Applications

(a) The singularities of every Green's function describing a planar domain of finite connectivity can always be transformed by a conformal mapping into logarithmic singularities.

(b) The logarithmic capacity [3] of the "holes" of the logarithmic domains can trivially be calculated.

(c) In some problems there is no distinction between conformal equivalent Laplace potentials. In these cases, the Green's function of the logarithmic domains allows for an analytical and computational treatment of all "really" different potentials.

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